Continuous selections, collectively fixed points and weakly T-KKM theorems in GFC-spaces

Phan Quoc Khanh

Department of Mathematics, International University of Ho Chi Minh City, Linh Trung, Thu Duc, Ho Chi Minh City, Viet Nam.

E-mail: pqkhanh@hcmiu.edu.vn

Vo Si Trong Long

Department of Mathematics, Cao Thang Technology College, Ho Chi Minh City, Viet Nam. E-mail: vstronglong@gmail.com

Nguyen Hong Quan

Department of Mathematics, Information Technology College of Ho Chi Minh City, Hoa Thanh, Tan Phu, Ho Chi Minh City, Viet Nam.

E-mail: nguyenhongquan1978@gmail.com

Abstract: We prove theorems on continuous selections, collectively fixed points, collectively coincidence points, weakly T-KKM mappings and minimax inequalities in GFC-spaces. Each of them is demonstrated by using its preceding assertions. Our results contain a number of existing ones in the recent literature.

Key words: Continuous selections, collectively fixed points and coincidence points, weakly KKM theorems, minimax inequalities, GFC-spaces

Mathematics subject classification: 49J40, 54C65, 47H04, 91B50, 54H25

1 Introduction

Continuous selection theorems play an important role in nonlinear analysis and applied mathematics. Since the first well-known result on continuous selections of [20], many international efforts have been made to develop sufficient conditions for the existence of continuous selections (and their applications) in increasingly general settings: paracompact spaces [4], C-spaces [13], G-convex spaces [21, 26],

L-convex spaces [6], C^∞-spaces [26] and FC-spaces [9].

Collectively fixed points along with collective coincidence points are natural extensions of fixed points, which have been applied in the existence study for almost all areas of mathematics. First result in this direction appeared in [24]. So far, this topic has been much developed under different assumptions and underlying spaces: convex subsets of topological vector spaces [1], L-convex spaces [6], G-convex spaces [7, 26] and FC-spaces [10]. In most of the mentioned papers, continuous selection theorems are the tool for the proofs and various applications are also discussed.

Another kind of existence theorems, which has also been attracted an increasing attention is KKM-type theorems. These theorems have been extended and improved in connection with generalizations of relaxed convex structures. In [12] a pure topological version of the classical KKM theorem was proposed replacing convex hulls by contract subsets. The above encountered G-convex space (introduced in [22]) and FC-space (proposed in [8]), among others, are continuations of this idea and also used to develop KKM-type theorems and related existence theorems. Very recently, in [11, 14-18] a GFC-space structure was proposed to include most of the above mentioned generalized convex settings and to investigate KKM-type theorems along with many related existence theorems like those on fixed points, coincidence points, saddle points, maximal elements, intersection points and alternative and minimax theorems. Furthermore, in [2, 23] weakly KKM theorems were established in G-convex spaces and FC-spaces.

However, we have not observed considerations which directly relate theorems on continuous selections or collectively fixed points to KKM-type theorems and KKM properties in general. Motivated by the above papers and this observation, the aim of this paper is to extend and improve some theorems of the mentioned three groups, namely theorems on continuous selections, collectively fixed and collective coincidence points, weakly KKM mappings and minimax inequalities, all in a GFC-space setting and in a close relation that each result is proved by its preceding results. Comparisons between our theorems and many previously existing ones are also provided.

The layout of the paper is as follows. In the rest of this section we recall the needed definitions. Section 2 is devoted to continuous selections and local continuous selections in underlying GFC-spaces. In Section 3 we establish theorems on collectively fixed points and collective coincidence points by using the continuous selection assertions of the preceding section. In the final Section 4 a weakly T-KKM theorem is demonstrated in GFC-spaces and applied to get minimax inequalities of Ky Fan's type.

We use only standard notations. If X is a topological space and $C, D \subseteq X$, $\mathrm{int}C$, $\mathrm{int}_D C$, and C^c stand for the interior, interior in D and complement of C, respectively. $\mathbb{N}$ and $\mathbb{R}$ denote the set of the natural numbers and that of the real numbers, respectively. $\langle A \rangle$ stands for the set of all nonempty finite subsets of a set A. For $n \in \mathbb{N}$, Δ_n stands for the n-simplex with the vertices being the unit vectors e_0, ..., e_n of a basis of R^{n+1}. Recall that a multivalued map $F : X \to Y$ between two topological spaces is called upper semicontinuous (shortly u.s.c.) at $x \in X$ if for any open subset $U \subseteq Y$ containing $F(x)$, there is a neighborhood V of x such that $F(x') \subseteq U$ for all $x' \in V$. A function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be u.s.c. at $x \in X$ if

$$\mathrm{limsup}_{x' \to x} f(x') \leq f(x).$$

Definition 1.1 (classical) Let X and Z be topological spaces, $G : Z \to 2^X$ be a multifunction.

(i) A (single-valued) continuous map $g : Z \to X$ is called a continuous selection of G if $g(z) \in G(z)$ for all $z \in Z$.

(ii) If, for each $z \in Z$ there are a neighborhood V of z and a (single-valued) continuous map $g : V \to X$ such that $g(z) \in G(z)$ for all $z \in V$, then G is said to be locally continuously selectionable.

Definition 1.2 ([14]) (i) Let X be a topological space, A be a nonempty set and Φ be a family of continuous mappings $\varphi : \Delta_n \to X, n \in \mathbb{N}$. Then a triple (X, A, Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) if for each finite subset $N = \{a_0, a_1, ..., a_n\}$ of A, there is $\varphi_N : \Delta_n \to X$ of the family Φ. (Later we also use $(X, A, \{\varphi_N\})$ to denote (X, A, Φ).)

(ii) Let $S : A \longrightarrow 2^X$ be a multivalued mapping. A subset D of A is called an S-subset of A if, for each $N = \{a_0, a_1, ..., a_n\} \subseteq A$ and each $\{a_{i_0}, a_{i_1}, ..., a_{i_k}\} \subseteq N \cap D$, one has $\varphi_N(\Delta_k) \subseteq S(D)$, where Δ_k is the face of Δ_n corresponding to $\{a_{i_0}, a_{i_1}, ..., a_{i_k}\}$, i.e. the simplex with vertices e_{i_0}, ..., e_{i_k}. (Roughly speaking, if D is an S-subset of A then $(S(D), D, \Phi)$ is a GFC-space.)

Definition 1.3 Let $(X$,A,$\Phi)$ be a GFC-space and Y be a nonempty set. Let $T : X \to 2^Y$, $F : A \to 2^Y$ be two set-valued mappings. F is called a weakly KKM mapping (KKM mapping, respectively) with respect to T, shortly, weakly T-KKM mapping (T-KKM mapping, respectively), if for each $N = \{a_0, ..., a_n\} \subseteq A$, each $\{a_{i_0}, ..., a_{i_k}\} \subseteq N$ and $x \in \varphi_N(\Delta_k)$, $T(x) \cap \bigcup_{j=0}^{k} F(a_{i_j}) \neq \emptyset$ ($T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} F(a_{i_j})$, respectively).

2 Continuous selection theorems

Theorem 2.1 *Let Z be a normal (topological) space, $(X, A, \{\varphi_N\})$ a GFC-space and $G : Z \to 2^X$. Assume that there is $F : Z \to 2^A$ such that the following conditions hold*

(i) *for each $z \in Z$, each $N = \{a_0, ..., a_n\} \subseteq A$ and each $\{a_{i_0}, ..., a_{i_k}\} \subseteq N \cap F(z)$ one has $\varphi_N(\Delta_k) \subseteq G(z)$, where Δ_k is the simplex formed by $e_{i_0}, ..., e_{i_k}$;*

(ii) *$Z = \bigcup_{i=0}^{m} \text{int}F^{-1}(\bar{a}_i)$ for some $\{\bar{a}_0, ..., \bar{a}_m\} \subseteq A$.*

Then G has a continuous selection g of the form $g = \varphi \circ \psi$ for some continuous maps $\varphi : \Delta_m \to X$ and $\psi : Z \to \Delta_m$.

Proof By (ii) and the normality of Z, there exists a continuous partition of unity $\{\psi_i\}_{i=0}^m$ of Z associated with the finite open cover $\{\text{int}F^{-1}(\bar{a}_i)\}_{i=0}^m$. Then, for each $z \in Z$, $i \in J(z) := \{j \in \{0, ..., m\} : \psi_j(z) \neq 0\}$ only if $z \in \text{int}F^{-1}(\bar{a}_i) \subseteq F(z)$, i.e. $\bar{a}_i \in F(z)$. Hence $\{\bar{a}_i : i \in J(z)\} \subseteq \{\bar{a}_0, ..., \bar{a}_m\} \cap F(z)$. On the other hand, due to the GFC-space structure, there is $\varphi_M : \Delta_M \to X$ associated with $\{\bar{a}_0, ..., \bar{a}_m\}$. Now we define $\psi : Z \to \Delta_m$ and $g : Z \to X$ as follows:

$$\psi(z) = \sum_{i=0}^{m} \psi_i(z)e_i,$$

$$g(z) = (\varphi_M \circ \psi)(z).$$

Then ψ and g are obviously continuous. Furthermore, for all $z \in Z$,

$$\sum_{i=0}^{m} \psi_i(z)e_i = \sum_{j \in J(z)} \psi_j(z)e_j \in \Delta_{J(z)}.$$

Hence, for all $z \in Z$,

$$g(z) = (\varphi_M \circ \psi)(z) \in \varphi_M(\Delta_{J(z)}) \subseteq G(z),$$

where the last inclusion is true by (i). Finally, putting $\varphi = \varphi_M$ we arrive at the conclusion.

□

Remark 1 Theorem 2.1 contains Theorem 2.1 of [10] as a special case for the FC-space setting and hence several earlier results like Proposition 1 of [5] and Theorem 2.2 of [25] for other space settings, etc. We will see in Example 2.1 below that the GFC-space setting of Theorem 2.1 not only generalizes the mentioned settings but also has real advantages. Furthermore, Theorem 2.1 improves Theorem 2.1 of [10] since Z is not necessary compact. This is also illustrated by Example 2.1.

Example 2.1 Let $Z = (0,4)$, $X = [0,+\infty)$ and $G : Z \to 2^X$ be defined by

$$G(z) = \begin{cases} 0 & \text{if } z \in (0,1], \\ [0,1] & \text{if } z \in (1,2], \\ [1,2] & \text{if } z \in (2,3), \\ [0,2] & \text{if } z \in [3,4). \end{cases}$$

Applying Theorem 2.1 for finding a continuous selection of G we choose $A = \mathbb{N}$ and, for $N = \{a_0, ..., a_n\} \in \langle A \rangle$, $\varphi_N : \Delta_n \to X$ defined by $\varphi_N(e) = \sum_{i=0}^n \lambda_i a_i$, where $e = \sum_{i=0}^n \lambda_i e_i \in \Delta_n$. Then, $(X, A, \{\varphi_N\})$ is a GFC-space. Next, we take $F : Z \to 2^A$ defined as follows

$$F(z) = \begin{cases} 0 & \text{if } z \in (0,1), \\ \{0,1\} & \text{if } z \in [1,2), \\ \{1,2\} & \text{if } z \in [2,3), \\ 2 & \text{if } z \in [3,4). \end{cases}$$

Then, assumption (i) of Theorem 2.1 is easily checked to be fulfilled. For (ii) we see that, for $\bar{a}_i = i, i = 0,1,2$,

$$Z = (0,2) \cup (1,3) \cup (2,4) = \bigcup_{i=0}^{2} \text{int} F^{-1}(\bar{a}_i).$$

By this theorem G has a continuous selection g. But Theorem 2.1 of [10] cannot be employed as Z is not compact. Now we compute g by finding $\psi : Z \to \Delta_2$ as φ_M corresponding to $\bar{a}_i = i, i = 0,1,2$ is already known. We can verify that a continuous partition of unity associated with the above cover of Z is $\{\psi_i\}_{i=0}^2 : Z \to [0,1]$ defined as follows:

$$\psi_0(z) = \begin{cases} 1 & \text{if } z \in (0,1], \\ 2 - z & \text{if } z \in (1,2), \\ 0 & \text{if elsewhere.} \end{cases}$$

$$\psi_1(z) = \begin{cases} z-1 & \text{if } z \in (1,2], \\ 3-z & \text{if } z \in (2,3), \\ 0 & \text{if elsewhere.} \end{cases}$$

$$\psi_2(z) = \begin{cases} z-2 & \text{if } z \in (2,3], \\ 1 & \text{if } z \in (3,4), \\ 0 & \text{if elsewhere.} \end{cases}$$

According to the proof of Theorem 2.1, for $z \in Z$,

$$\psi(z) = \sum_{i=0}^{2} \psi_i(z) e_i = (\psi_0(z), \psi_1(z), \psi_2(z))$$

and hence g is defined by

$$g(z) = (\varphi_M \circ \psi)(z) = \begin{cases} 0 & \text{if } z \in (0,1], \\ z-1 & \text{if } z \in (1,3), \\ 2 & \text{if } z \in [3,4). \end{cases}$$

We can also verify directly that g is a continuous selection of G.

Condition (ii) of Theorem 2.1 can be modified as follows.

Theorem 2.2 *Let Z be normal, $(X, A, \{\varphi_N\})$ a GFC-space and $G : Z \to 2^X$. Assume that there is $F : Z \to 2^A$ such that the following conditions hold*

(i) *for each $z \in Z$, each $N = \{a_0, ..., a_n\} \subseteq A$ and each $\{a_{i_0}, ..., a_{i_k}\} \subseteq N \cap F(z)$ one has $\varphi_N(\Delta_k) \subseteq G(z)$;*

(ii$_1$) *for each nonempty compact subset K of Z, one has $K \subseteq \bigcup_{a \in A} \text{int} F^{-1}(a)$;*

(ii$_2$) *there exists a finite subset $\{\bar{a}_0, ..., \bar{a}_m\}$ of A such that either of the following statements holds*

(ii$_2^1$) $\bigcap_{i=0}^{m} (\text{int} F^{-1}(\bar{a}_i))^c$ *is nonempty and compact;*

(ii$_2^2$) $Z \setminus K \subseteq \bigcup_{i=1}^{m} \text{int} F^{-1}(\bar{a}_i)$ *for some nonempty compact subset K of Z.*

Then G has a continuous selection g of the form $g = \varphi \circ \psi$, where $\varphi : \Delta_m \to X$ and $\psi : Z \to \Delta_m$ are continuous.

Proof Note that (ii$_2^1$) implies (ii$_2^2$). Hence we need to prove the theorem only for the latter. In view of (ii$_1$), for the compact subset K provided by (ii$_2^2$), there exists a finite subset $\{\hat{a}_0, ..., \hat{a}_l\}$ of A such that $K \subseteq \bigcup_{i=0}^{l} \mathrm{int}F^{-1}(\hat{a}_i)$. Set $\{\tilde{a}_0, ..., \tilde{a}_h\} := \{\bar{a}_0, ..., \bar{a}_m, \hat{a}_0, ..., \hat{a}_l\}$ we have

$$Z = (Z \setminus K) \cup K = \bigcup_{i=0}^{h} \mathrm{int}F^{-1}(\tilde{a}_i).$$

Thus, the assumption (ii) of Theorem 2.1 is fulfilled and we are done. □

Remark 2 (i) Theorem 2.2 generalizes Theorem 2.1 of [6], Theorem 3.1 of [7], Theorem 2.2 of [9], and Theorem 1 of [26], since being an L-convex space or G-convex space is a special case of being a GFC-space and each paracompact space is also a normal space.

(ii) It is known that each compact Hausdorff space and each paracompact space are normal. If Z is a compact Hausdorff space, of course assumption (ii) of Theorem 2.1 (assumptions (ii_1)-(ii_2) of Theorem 2.2) can be replaced by the weaker condition that $Z = \bigcup_{a \in A} \mathrm{int}F^{-1}(a)$. For paracompact spaces, Theorem 2.3 below asserts only the existence of a local continuous selection under this weaker condition.

Theorem 2.3 *Suppose Z is a paracompact space, $(X, A, \{\varphi_N\})$ a GFC-space and $G : Z \to 2^X$. Assume that there is $F : Z \to 2^A$ such that*

(i) *for all $z \in Z$, all $N = \{a_0, ..., a_n\} \subseteq A$ and all $\{a_{i_0}, ..., a_{i_k}\} \subseteq N \cap F(z)$, $\varphi_N(\Delta_k) \subseteq G(z)$;*

(ii) *$Z = \bigcup_{a \in A} \mathrm{int}F^{-1}(a)$.*

Then G is locally continuously selectionable.

Proof Let $z \in Z$ be arbitrary. By the paracompactness of Z, there is a locally finite refinement $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of the open cover $\{\mathrm{int}F^{-1}(a)\}_{a \in A}$ of Z provided by (ii). Hence, for the given z, there exist its open neighborhood V and $M := \{a_{\lambda_0}, ..., a_{\lambda_m}\} \subseteq A$ such that $\emptyset \neq V \cap U_{\lambda_i} \subseteq \mathrm{int}F^{-1}(a_{\lambda_i})$ for $i = 1, ..., m$. Corresponding to M we have $\varphi_M : \Delta_m \to X$. As any paracompact space, Z is normal and hence there is a continuous partition of unity $\{\psi_\lambda\}_{\lambda \in \Lambda}$ associated with the cover $\mathcal{U}$. Define $g : V \to X$ by

$$g(z') = \varphi_M\Big(\sum_{i=0}^{m} \psi_{\lambda_i}(z') e_i\Big).$$

Then, g is continuous. Furthermore, $j \in J(z') := \{j \in \{0, ..., m\} : \psi_{\lambda_j}(z') \neq 0\}$ only if

$$z' \in U_{\lambda_j} \subseteq \mathrm{int}F^{-1}(a_{\lambda_j}) \subseteq F^{-1}(a_{\lambda_j}).$$

Therefore, for all $z' \in V$, $\{a_{\lambda_j} : j \in J(z')\} \subseteq F(z')$. Finally we have

$$g(z') = \varphi_M\Big(\sum_{i=0}^{m} \psi_{\lambda_i}(z')e_i\Big) \in \varphi_M(\Delta_{J(z')}) \subseteq G(z')$$

(the last inclusion is due to (i)). Thus, g is a continuous selection of $G|_V$. □

Remark 3 Theorem 2.3 improves Theorem 4 of [21], where the underlying space is a G-convex space instead of a GFC-space. Observe that for a G-convex structure to be defined we need a set-valued map $\Gamma :< A >\to 2^X$ and a family of continuous maps $\Phi_N : \Delta_n \to \Gamma(N)$ such that $\Phi_N(\Delta_k) \subseteq \Gamma(N_k)$ for each $N_k \subseteq N$, while for a GFC-space only a family of continuous maps $\Phi_N : \Delta_n \to \Gamma(N)$ (without additional conditions) is needed.

3 Collectively fixed points and collective coincidence points

Applying of the above results, we establish some existence theorems for collectively fixed points and collective coincidence points. We precise our problems as follows. Let I be an index set, X_i, $i \in I$, be topological spaces, $X = \prod_{i\in I} X_i$ (here and in the sequel all products of topological spaces are Tikhonov products) and $G_i : X \to 2^{X_i}$ be given multifunctions. The collectively fixed point problem is of

(CFP) finding $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x}_i \in G_i(\bar{x})$ for all $i \in I$.

Let J be another index set, Y_j, $j \in J$, be topological spaces, $Y = \prod_{j\in J} Y_j$, $G_j : X \to 2^{Y_j}$ and $H_i : Y \to 2^{X_i}$ be given. Our collective coincidence point problem is of

(CPP) finding $(\bar{x}, \bar{y}) \in X \times Y$ such that

$\bar{x}_i \in H_i(\bar{y})$ and $\bar{y}_j \in G_j(\bar{x})$ for all $(i, j) \in I \times J$.

Theorem 3.1 *For problem* (CFP)*, suppose that X is a normal space and $(X_i, A_i, \{\varphi_{N_i}\})_{i\in I}$ is a family of GFC-spaces. Assume that, for each $i \in I$, there exists $F_i : X \to 2^{A_i}$ such that the following conditions hold*

(i) *for each $x \in X$, each $N_i := \{a^i_0, ..., a^i_{n_i}\} \subseteq A_i$ and each $\{a^i_{j_0}, ..., a^i_{j_{k_i}}\} \subseteq N_i \cap F_i(x)$, one has $\varphi_{N_i}(\Delta_{k_i}) \subseteq G_i(x)$ for all $i \in I$;*

(ii) *for all $i \in I$, there exists a finite subset $\{\bar{a}^i_0, ..., \bar{a}^i_{m_i}\}$ of A_i such that one of the following two conditions holds* (ii$_1$) *$X = \bigcup_{j=0}^{m_i} \mathrm{int} F_i^{-1}(\bar{a}_j^i)$ for all $i \in I$;*

(ii$_2$) *for each compact subset K of X, one has $K \subseteq \bigcup_{a^i \in A_i} \mathrm{int} F_i^{-1}(a^i)$, and either the set $\bigcap_{j=0}^{m_i} (\mathrm{int} F_i^{-1}(\bar{a}_j^i))^c$ is nonempty and compact (for each i); or there is a nonempty compact subset K_i of X satisfying $X \setminus K_i \subseteq \bigcup_{j=0}^{m_i} \mathrm{int} F_i^{-1}(\bar{a}_j^i)$ for each i.*

Then (CFP) *has solutions.*

Proof For each $i \in I$, by Theorems 2.1 and 2.2, G_i has a continuous selection g_i, and there are continuous maps $\varphi_i : \Delta_{m_i} \to X_i$ and $\psi_i : X \to \Delta_{m_i}$ such that $g_i = \varphi_i \circ \psi_i$. Of course $\Delta := \prod_{i \in I} \Delta_{m_i}$ is a compact convex subset of $\mathbb{R}^I := \prod_{i \in I} \mathbb{R}^{m_i+1}$. Let $p_i : \Delta \to \Delta_{m_i}$ be the canonical projection of Δ onto Δ_{m_i}. We define two mappings $\Omega : \Delta \to X$ and $\Psi : X \to \Delta$ by

$$\Omega(t) = \prod_{i \in I} \varphi_i(p_i(t)) \text{ for all } t \in \Delta,$$

$$\Psi(x) = \prod_{i \in I} \psi_i(x) \text{ for all } x \in X.$$

Then, Ω and Ψ are continuous and so is $\Psi \circ \Omega : \Delta \to \Delta$. By virtue of the Tikhonov fixed-point theorem, there exists $\bar{t} \in \Delta$ such that $(\Psi \circ \Omega)(\bar{t}) = \bar{t}$. Setting $\bar{x} = \Omega(\bar{t})$ we have

$$\begin{aligned} \bar{x} &= \Omega(\Psi(\bar{x})) \\ &= \Omega\big(\textstyle\prod_{i \in I} \psi_i(\bar{x})\big) \\ &= \textstyle\prod_{i \in I} \varphi_i\Big(p_i\big(\prod_{i \in I} \psi_i(\bar{x})\big)\Big) \\ &= \textstyle\prod_{i \in I} (\varphi_i \circ \psi_i)(\bar{x}). \end{aligned}$$

It follows that $\bar{x}_i = (\varphi_i \circ \psi_i)(\bar{x}) = g_i(\bar{x}) \in G_i(\bar{x})$ for all $i \in I$. Hence, $\bar{x}$ is a solution of (CFP). □

Remark 4 (i) Similarly as in Remark 2(ii), if X is compact and Hausdorff, assumption (ii) of Theorem 3.1 can be reduced to the weaker condition that $X = \bigcup_{a^i \in A_i} \mathrm{int} F_i^{-1}(\bar{a}^i)$ for all $i \in I$.

(ii) Theorem 3.1 generalizes and improves Theorem 3 of [21] (with a G-space setting and all X_i being Hausdorff and compact). Note also that when applied to the particular case of G-space settings, Theorem 3.1 collapses to Theorem 7 of [26] and if, in addition $A_i \equiv X_i$ in the G-convex spaces involved, Theorem 3.1 contains Theorems 3.1 and 3.2 of [7].

When X is Hausdorff, without compactness or normality assumptions, problem (CFP) still has solutions with a modification of assumption (ii) as follows.

Theorem 3.2 *For problem* (CFP)*, suppose that X is Hausdorff, $(X_i, A_i, \{\varphi_{N_i}\})_{i\in I}$ is a family of GFC-spaces and, for each $i \in I$, there exists $F_i : X \to 2^{A_i}$ with the following properties*

(i) *for each $x \in X$, each $N_i = \{a^i_0, ..., a^i_{n_i}\} \subseteq A_i$ and each $\{a^i_{j_0}, ..., a^i_{j_{k_i}}\} \subseteq N_i \cap F_i(x)$, $\varphi_{N_i}(\Delta_{k_i}) \subseteq G_i(x)$;*

(ii$'$) *for each compact subset K of X and each $i \in I$, $K \subseteq \bigcup_{a^i\in A_i} \mathrm{int}F_i^{-1}(a^i)$;*

(ii$''$) *for each $i \in I$, there exist a multimap $S_i : A_i \to 2^{X_i}$, a nonempty subset A^0_i of A_i such that $\bigcap_{a^i\in A^0_i}(\mathrm{int}F_i^{-1}(a^i))^c$ is compact or empty and that, for each finite subset M_i of A_i, there is an S_i-subset L_{M_i} of A_i, containing $A^0_i \cup M_i$, with $S_i(L_{M_i})$ being compact.*

Then (CFP) *has a solution.*

Proof For $i \in I$ setting $K_i := \bigcap_{a^i\in A^0_i}(\mathrm{int}F_i^{-1}(a^i))^c$ we have

$$X \setminus K_i = \bigcup_{a^i\in A^0_i} \mathrm{int}F_i^{-1}(a^i).$$

As K_i is compact, by (ii') $K_i \subseteq \bigcup_{a^i\in A_i} \mathrm{int}F_i^{-1}(a^i)$, and hence there is a finite subset M_i of A_i such that $K_i \subseteq \bigcup_{a^i\in M_i} \mathrm{int}F_i^{-1}(a^i)$. Then, if K_i is nonempty,

$$X = (X \setminus K_i) \cup K_i = \bigcup_{a^i\in A^0_i\cup M_i} \mathrm{int}F_i^{-1}(a^i).$$

If K_i is empty then, for all finite subset M_i of A_i,

$$X = \bigcup_{a^i\in A^0_i} \mathrm{int}F_i^{-1}(a^i) = \bigcup_{a^i\in A^0_i\cup M_i} \mathrm{int}F_i^{-1}(a^i).$$

Thus, the equality occurs for the finite subset M_i of A_i and all $i \in I$. Hence

$$X = \bigcup_{a^i\in L_{M_i}} \mathrm{int}F_i^{-1}(a^i). \tag{3.1}$$

Observe that the family $\{(S_i(L_{M_i}), L_{M_i}, \varphi_{N_i})\}_{i\in I}$ is a family of GFC-spaces. We set $X_M = \prod_{i\in I} S_i(L_{M_i})$ (then X_M is compact and Hausdorff and hence normal) and, for each $i \in I$, define two new mappings $\hat{G}_i : X_M \to 2^{S_i(L_{M_i})}$ and $\hat{F}_i : X_M \to 2^{L_{M_i}}$ as follows

$$\hat{G}_i(x) = G_i(x) \cap S_i(L_{M_i}),$$

$$\hat{F}_i(x) = F_i(x) \cap L_{M_i}.$$

We check assumptions (i) - (ii$_1$) of Theorem 3.1 for $\hat{G}_i$ and $\hat{F}_i$. By (i) and the definition of S_i-subsets, for each $x \in X_M$, each $N_i = \{a^i_0, ..., a^i_{n_i}\} \subseteq L_{M_i}$ and each $\{a^i_{j_0}, ..., a^i_{j_{k_i}}\} \subseteq N_i \cap \hat{F}_i(x) = N_i \cap F_i(x) \cap L_{M_i}$, we have

$$\varphi_{N_i}(\Delta_{k_i}) \subseteq G_i(x) \cap S_i(L_{M_i}) = \hat{G}_i(x)$$

as required in assumption (i) of Theorem 3.1. For (ii$_1$), by (3.1) we have, for each $i \in I$,

$$X_M = \Big(\bigcup\nolimits_{a^i \in L_{M_i}} \mathrm{int} F_i^{-1}(a^i)\Big) \cap X_M = \bigcup\nolimits_{a^i \in L_{M_i}} \mathrm{int}_{X_M}\Big(F_i^{-1}(a^i) \cap X_M\Big).$$

On the other hand, for all $a^i \in L_{M_i}$,

$$\hat{F}_i^{-1}(a^i) = \{x \in X \mid a^i \in F_i(x)\} \cap X_M = F_i^{-1}(a^i) \cap X_M.$$

Hence $X_M = \bigcup_{a^i \in L_{M_i}} \mathrm{int}_{X_M} \hat{F}^{-1}(a^i)$. Since X_M is compact, there exists a finite subset $\{\bar{a}^i_0, ..., \bar{a}^i_{m_i}\}$ of L_{M_i} satisfying $X_M = \bigcup_{j=0}^{m_i} \mathrm{int}_{X_M} \hat{F}^{-1}(\bar{a}^i_j)$, i.e. (ii$_1$) of Theorem 4.1 is satisfied. According to this theorem, problem (CFP) for $\{\hat{G}_i\}_{i \in I}$ has a solution, which is also a solution of problem (CFP) for $\{G_i\}_{i \in I}$. □

Remark 5 (i) Note that, for a topological space X, a set A and $F : X \to 2^A$, if $F(x) \neq \emptyset$ for all $x \in X$ and $F^{-1}(a)$ is open for all $a \in A$, then $X = \bigcup_{a \in A} \mathrm{int} F^{-1}(a)$, but the converse is not true (see e.g. Example 2.1 in the preceding section). Theorem 2.1 of [19] asserts the existence of a collectively fixed point for problem (CFP) for convex sets and topological vector spaces, a special case of GFC-spaces. Applied to this special case, Theorem 3.1 sharpens that Theorem 2.1 since we require the latter (weaker condition) in the above implication and that theorem imposes the former.

(ii) Theorem 3.2 includes Theorem 3.4 for underlying FC-spaces of [9], Theorem 3 with a G-convex structure of [21], Theorem 1 for convex subsets of topological vector spaces in [1], since our GFC-space setting contains all these structures.

The following obvious corollary about a fixed point will be used to prove a weakly T-KKM theorem in Section 4.

Corollary 3.1 *Let X be a compact Hausdorff space, $(X, A, \{\varphi_N\})$ a GFC-space and $G : X \to 2^X$. Suppose that there exists $F : X \to 2^A$ satisfying the following conditions*

(i) *for each* $x \in X$*, each* $N = \{a_0, ..., a_n\} \subseteq A$ *and each* $\{a_{i_0}, ..., a_{i_k}\} \subseteq N \cap F(x)$*,* $\varphi_N(\Delta_k) \subseteq G(x)$*;*

(ii) *either of the following two conditions* (ii_1)- (ii_2) *holds*

(ii_1) *there exists a finite subset* $\{\bar{a}_0, ..., \bar{a}_m\}$ *of* A *such that* $X = \bigcup_{i=0}^{m} \mathrm{int}F^{-1}(\bar{a}_i)$*;*

(ii_2) *for each compact subset* K *of* X*,* $K \subseteq \bigcup_{a \in A} \mathrm{int}F^{-1}(a)$*, and one of the following three statements is true*

(ii_2^1) *there exists a finite subset* $\{\bar{a}_0, ..., \bar{a}_m\}$ *of* A *such that the set* $\bigcap_{i=0}^{m}(\mathrm{int}F^{-1}(\bar{a}_i))^c$ *is nonempty and compact;*

(ii_2^2) *there are a finite subset* $\{\bar{a}_0, ..., \bar{a}_m\}$ *of* A *and a nonempty compact subset* K *of* X *with* $X \setminus K \subseteq \bigcup_{i=0}^{m} \mathrm{int}F^{-1}(\bar{a}_i)$*;*

(ii_2^3) *there exist a multimap* $S : A \to 2^X$*, a nonempty subset* A_0 *of* A *such that* $\bigcap_{a \in A_0}(\mathrm{int}F^{-1}(a))^c$ *is compact or empty and that, for each finite subset* M *of* A*, there is an* S*-subset* L_M *of* A *containing* $A_0 \cup M$ *with* $S(L_M)$ *being compact.*

Then there exists $\bar{x} \in X$ *such that* $\bar{x} \in G(\bar{x})$*.*

Now we pass to problem (CPP) about collective coincidence points.

Theorem 3.3 *For problem* (CPP)*, suppose that* X *is a normal space,* $\{X_j, B_j, \{\psi_{N_j}\})_{j \in J}$ *and* $(Y_i, A_i, \{\varphi_{N_i}\})_{i \in I}$ *are two families of GFC-spaces and, for each* $(i, j) \in I \times J$*, there exist* $F_j : X \to 2^{B_j}$ *and* $T_i : Y \to 2^{A_i}$ *with the following properties*

(i) *for each* $x \in X$*, each* $N_j = \{b_0^j, ..., b_{n_j}^j\} \subseteq B_j$ *and each* $\{b_{l_0}^j, ..., b_{l_{k_j}}^j\} \subseteq N_j \cap F_j(x)$*,* $\psi_{N_j}(\Delta_{k_j}) \subseteq G_j(x)$*;*

(ii) *for each* $y \in Y$*, each* $N_i = \{a_0^i, ..., a_{n_i}^i\} \subseteq A_i$ *and each* $\{a_{l_0}^i, ..., a_{l_{k_i}}^i\} \subseteq N_i \cap T_i(y)$*,* $\varphi_{N_i}(\Delta_{k_i}) \subseteq H_i(y)$*;*

(iii) *there are finite subsets* $\{\bar{a}_0^i, ..., \bar{a}_{m_i}^i\}$ *of* A_i *and* $\{\bar{b}_0^j, ..., \bar{b}_{m_j}^j\}$ *of* B_j *such that* $X = \bigcup_{l=0}^{m_j} \mathrm{int}F_j^{-1}(\bar{b}_l^j)$ *and* $Y = \bigcup_{l=0}^{m_i} \mathrm{int}T_i^{-1}(\bar{a}_l^i)$*;*

Then there exist solutions of (CPP) .

Proof For each $j \in J$, by (i), (iii) and Theorems 2.1, G_j has a continuous selection $g_j : X \to Y_j$ and hence we obtain a continuous map $g : X \to Y$ defined by $g(x) = \prod_{j \in J} g_j(x)$. For each $i \in I$, define two new multimaps $Q_i : X \to 2^{X_i}$ and $P_i : X \to 2^{A_i}$ by setting

$$Q_i(x) = H_i(g(x)),$$

$$P_i(x) = T_i(g(x)).$$

We see from (ii) that, for each $x \in X$, each $N_i = \{a_0^i, ..., a_{n_i}^i\} \subseteq A_i$ and each $\{a_{l_0}^i, ..., a_{l_{k_i}}^i\} \subseteq N_i \cap P_i(x) = N_i \cap T_i(g(x))$,

$$\varphi_{N_i}(\Delta_{k_i}) \subseteq H_i(g(x)) = Q_i(x).$$

while from (iii) we obtain a finite subset $\{\bar{a}_0^i, ..., \bar{a}_{m_i}^i\}$ of A_i such that

$$Y = \bigcup_{l=0}^{m_i} \mathrm{int} T_i^{-1}(\bar{a}_l^i).$$

It follow that

$$\begin{aligned} X &= g^{-1}(Y) \\ &= g^{-1}\Big(\textstyle\bigcup_{l=0}^{m_i} \mathrm{int} T_i^{-1}(\bar{a}_l^i)\Big) \\ &= \textstyle\bigcup_{l=0}^{m_i} g^{-1}\Big(\mathrm{int} T_i^{-1}(\bar{a}_l^i)\Big) \\ &= \textstyle\bigcup_{l=0}^{m_i} \mathrm{int}\Big(g^{-1}(T_i^{-1}(\bar{a}_l^i))\Big) \\ &= \textstyle\bigcup_{l=0}^{m_i} \mathrm{int} P_i^{-1}(\bar{a}_l^i). \end{aligned}$$

Now that the assumptions of Theorem 3.1 hold for Q_i and P_i, it yields an $\bar{x} \in X$ such that, for all $i \in I$,

$$\bar{x}_i \in Q_i(\bar{x}) = H_i(g(\bar{x})).$$

Setting $\bar{y} = g(\bar{x})$ one sees that, for all $(i, j) \in I \times J$, $\bar{x}_i \in H_i(\bar{y})$ and $\bar{y}_j \in G_j(\bar{x})$. Thus, $(\bar{x}, \bar{y})$ is a solution of (CPP). □

Remark 6 Theorem 3.5 of [9] for an FC-space structure and Theorem 8 of [26] for convex subsets of topological vector space setting are special cases of Theorem 3.3 (in [26] additional compactness assumptions are imposed).

Without the normality of X but with X_i being Hausdorff topological spaces for all $i \in I$, we can modify assumption (iii) in Theorem 3.3 to keep the solvability of (CPP) as follows.

Theorem 3.4 *For problem* (CPP), *suppose that* $X_j, B_j, \psi_{N_j})_{j\in J}$ *and* $(Y_i, A_i, \varphi_{N_i})_{i\in I}$ *are two families of GFC-spaces and, for each* $(i, j) \in I \times J$, X_i *is a Hausdorff topological space. Assume further that there exist* $F_j : X \to 2^{B_j}$ *and* $T_i : Y \to 2^{A_i}$ *possessing the following properties*

(i) *for each* $j \in J$*, each* $x \in X$*, each* $N_j = \{b^j_0, ..., b^j_{n_j}\} \subseteq B_j$ *and each* $\{b^j_{l_0}, ..., b^j_{l_{k_j}}\} \subseteq N_j \cap F_j(x)$ *one has* $\psi_{N_j}(\Delta_{k_j}) \subseteq G_j(x)$*;*

(ii) *for each* $i \in I$*, each* $y \in Y$*, each* $N_i = \{a^i_0, ..., a^i_{n_i}\} \subseteq A_i$ *and each* $\{a^i_{l_0}, ..., a^i_{l_{k_i}}\} \subseteq N_i \cap T_i(y)$*,* $\varphi_{N_i}(\Delta_{k_i}) \subseteq H_i(y)$*;*

(iii′) *for each* $(i, j) \in I \times J$*,* $X = \bigcup_{b^j \in B_j} \mathrm{int}F_j^{-1}(b^j)$ *and* $Y = \bigcup_{a^i \in A_i} \mathrm{int}T_i^{-1}(a^i)$*;*

(iii″) *there exist a multimap* $S_i : A_i \to 2^{X_i}$*, a nonempty subset* A^0_i *of* A_i *such that* $\bigcap_{a^i \in A^0_i} (\mathrm{int}T_i^{-1}(a^i))^c$ *is compact or empty and that, for each finite subset* M_i *of* A_i*, there is an* S_i*-subset* L_{M_i} *of* A_i *containing* $A^0_i \cup M_i$ *with* $S_i(L_{M_i})$ *being compact.*

Then (CPP) *has a solution.*

Proof For $i \in I$ setting $K^i_Y := \bigcap_{a^i \in A^0_i} (\mathrm{int}T_i^{-1}(a^i))^c$ we have

$$Y \setminus K^i_Y = \bigcup_{a^i \in A^0_i} \mathrm{int}T_i^{-1}(a^i).$$

As K^i_Y is compact and has open cover $\{\mathrm{int}T_i^{-1}(a^i)\}_{a^i \in A_i}$ by (iii'), there is a finite subset M_i of A_i such that $K^i_Y \subseteq \bigcup_{a^i \in M_i} \mathrm{int}T_i^{-1}(a^i)$. Then, if K^i_Y is nonempty,

$$Y = (Y \setminus K^i_Y) \cup K^i_Y = \bigcup_{a^i \in A^0_i \cup M_i} \mathrm{int}T_i^{-1}(a^i),$$

while if K^i_Y is empty, for any finite subset M_i of A_i one has

$$Y = \bigcup_{a^i \in A^0_i} \mathrm{int}T_i^{-1}(a^i) = \bigcup_{a^i \in A^0_i \cup M_i} \mathrm{int}T_i^{-1}(a^i).$$

Therefore, Y has an open cover:

$$Y = \bigcup_{a^i \in A^0_i \cup M_i} \mathrm{int}T_i^{-1}(a^i)$$

and hence, by (iii”),

$$Y = \bigcup_{a^i \in L_{M_i}} \mathrm{int}T_i^{-1}(a^i). \tag{3.2}$$

On the other hand, observe that $\{(S_i(L_{M_i}), L_{M_i}, \varphi_{N_i})\}_{i \in I}$ is a family of GFC-spaces and $X_M := \prod_{i \in I} S_i(L_{M_i})$ is compact and Hausdorff hence a normal space. For each $j \in J$, to apply Theorem 2.1 for $\hat{G}_j := G_j|_{X_M} : X_M \to 2^{Y_j}$ and $\hat{F}_j := F_j|_{X_M} : X_M \to 2^{B_j}$ we see that its assumption (i) is satisfied by (i) of Theorem 4.4 and the definition of S_i-subsets. For (ii) we have

$$X_M \subseteq \bigcup_{b^j \in B_j} \mathrm{int}F_j^{-1}(b^j) \cap X_M$$

$$= \bigcup_{b^j \in B_j} \text{int}_{X_M}(F_j^{-1}(b^j) \cap X_M)$$
$$= \bigcup_{b^j \in B_j} \text{int}_{X_M} \hat{F}_j^{-1}(b^j).$$

Due to the compactness of X_M, from this we can extract a finite cover as (ii) desires. Thus, $\hat{G}_j$ has a continuous selection $g_j : X_M \to 2^{Y_j}$ and so we obtain a continuous map $g : X_M \to 2^Y$ defined by $g(x) = (g_j(x))_{j \in J}$. For each $i \in I$, define two new mappings $Q_i : X_M \to 2^{S_i(L_{M_i})}$ and $P_i : X_M \to 2^{L_{M_i}}$ as follows

$$Q_i(x) = H_i(g(x)) \cap S_i(L_{M_i}),$$

$$P_i(x) = T_i(g(x)) \cap L_{M_i}.$$

Now we verify assumptions (i) and (ii$_1$) of Theorem 3.1 for Q_i and P_i. Similarly as checking (i) of Theorem 2.1 but now by (ii), we have, for each $x \in X_M$, each $N_i = \{a_0^i, ..., a_{n_i}^i\} \subseteq L_{M_i}$ and each $\{a_{l_0}^i, ..., a_{l_{k_i}}^i\} \subseteq N_i \cap P_i(x) = N_i \cap T_i(g(x)) \cap L_{M_i}$,

$$\varphi_{N_i}(\Delta_{k_i}) \subseteq H_i(g(x)) \cap S_i(L_{M_i}) = Q_i(x)$$

as required. Now for (ii$_1$) of Theorem 3.1, by (3.2) we have, for each $i \in I$,

$$\begin{aligned} X_M &= g^{-1}(Y) \\ &= g^{-1}\Big(\textstyle\bigcup_{a^i \in L_{M_i}} \text{int}_Y T_i^{-1}(a^i)\Big) \\ &= \textstyle\bigcup_{a^i \in L_{M_i}} g^{-1}\Big(\text{int}_Y T_i^{-1}(a^i)\Big) \\ &= \textstyle\bigcup_{a^i \in L_{M_i}} \text{int}_{X_M}\Big(g^{-1}(T_i^{-1}(a^i))\Big) \\ &= \textstyle\bigcup_{a^i \in L_{M_i}} \text{int}_{X_M} P_i^{-1}(a^i) \end{aligned}$$

and the compactness gives from this a finite cover for X_M as (ii$_1$) requires. According to Theorem 3.1 an $\bar{x} \in X_M \subseteq X$ exists such that, for all $i \in I$,

$$\bar{x}_i \in Q_i(\bar{x}) = H_i(g(\bar{x})) \cap S_i(L_{M_i}) \subseteq H_i(g(\bar{x})).$$

Set $\bar{y} = g(\bar{x}) = (g_j(\bar{x}))_{j \in J}$. Then, $\bar{x}_i \in H_i(\bar{y})$ and $\bar{y}_j \in \hat{G}_j(\bar{x}) \subseteq G_j(\bar{x})$ for all $(i, j) \in I \times J$. Thus, $(\bar{x}, \bar{y})$ is a solution of (CPP). □

Remark 7 Theorem 3.4 contains Theorem 3.1 of [10] for a FC-space setting and Theorem 3.6 of [9] for a convex subset setting as special cases.

4 Weakly T-KKM theorems and minimax inequalities

In this section, using results in Section 3 we establish a weakly T-KKM theorem and then apply it to minimax inequalities.

Theorem 4.1 *Let X be a Hausdorff space, $(X, A, \{\varphi_N\})$ a GFC-space, Y a nonempty set, $T : X \to 2^Y$ and $H : A \to 2^Y$. Assume that*

(i) *H is a weakly T-KKM mapping;*

(ii) *for each $a \in A$, the set $\{x \in X : T(x) \cap H(a) \neq \emptyset\}$ is closed.*

Then the following statements hold

(i) *if, additionally, X is compact, then there exists a point $\bar{x} \in X$ such that $T(\bar{x}) \cap H(a) \neq \emptyset$ for each $a \in A$;*

(ii) *for each finite subset $N = \{a_0, ..., a_n\}$ of A, there exists a point $\bar{x} \in \varphi_N(\Delta_n)$ such that $T(\bar{x}) \cap H(a_i) \neq \emptyset$ for each $i \in \{0, 1, ..., n\}$.*

Proof (i) Reasoning ad absurdum, suppose, for each $x \in X$, there exits $a \in A$ such that $T(x) \cap H(a) = \emptyset$. Define $F : X \to 2^A$ and $G : X \to 2^X$ by

$$F(x) = \{a \in A \mid T(x) \cap H(a) = \emptyset\},$$

$$G(x) = \{x' \in X \mid \exists a \in F(x), T(x') \cap H(a) \neq \emptyset\}.$$

We use Corollary 3.1 for G and F. Clearly F has nonempty values. For each $a \in A$, $F^{-1}(a) = \{x \in X : T(x) \cap H(a) = \emptyset\}$ is open by (ii). Hence $X = \bigcup_{a \in A} \mathrm{int} F^{-1}(a)$ and then assumption (ii$_1$) of Corollary 3.1 is satisfied by the compactness. Furthermore, G has no fixed point. Indeed, if $x \in G(x)$ then there is $a \in F(x)$ such that $T(x) \cap H(a) \neq \emptyset$ contradicting the definition of F. Therefore, assumption (i) of Corollary 3.1 must be violated, i.e. there are $\hat{x} \in X$, $\bar{N} = \{\bar{a}_0, ..., \bar{a}_{\bar{n}}\} \subseteq A$ and $\bar{N}_k = \{\bar{a}_{i_0}, ..., \bar{a}_{i_k}\} \subseteq \bar{N} \cap F(\hat{x})$ such that $\varphi_{\bar{N}}(\Delta_k) \nsubseteq G(\hat{x})$. Then there must be an $\bar{x} \in \varphi_{\bar{N}}(\Delta_k)$ which does not belong to $G(\hat{x})$, i. e. for each $a \in F(\hat{x})$ we have $T(\bar{x}) \cap H(a) = \emptyset$. Hence $T(\bar{x}) \cap H(\bar{N}_k) = \emptyset$. On the other hand, since H is a weakly T-KKM mapping and $\bar{x} \in \varphi_{\bar{N}}(\Delta_k)$, we get $T(\bar{x}) \cap H(\bar{N}_k) \neq \emptyset$. By this contradiction we are done.

(ii) Let $N = \{a_0, ..., a_n\}$ be any finite subset of A. Each $M = \{a_{i_0}, ..., a_{i_m}\} \subseteq N$ corresponds to the continuous map $\varphi_M \equiv \varphi_N|_{\Delta_m} : \Delta_m \to \varphi_N(\Delta_n)$. Therefore, $(\varphi_N(\Delta_n), N, \{\varphi_M\})$ is a GFC-space. Set $\hat{H} := H|_N$ and $\hat{T} := T|_{\varphi_N(\Delta_n)}$. We claim that $\hat{H}$ is a weakly $\hat{T}$-KKM mapping. Indeed, for all $M = \{a_{i_0}, ..., a_{i_m}\} \subseteq N$, all $M_k \subseteq M$ and all $x \in \varphi_M(\Delta_k) \subseteq \varphi_N(\Delta_k)$, one has $\hat{T}(x) \cap \hat{H}(M_k) = T(x) \cap H(M_k) \neq \emptyset$ (as H is a weakly T-KKM mapping).

As $\varphi_N(\Delta_n)$ is compact, applying statement (i) to GFC-space $(\varphi_N(\Delta_n), N, \{\varphi_M\})$ with H, T replaced by $\hat{H}$, $\hat{T}$ we complete the proof. $\square$

Remark 8 (i) Theorem 2 of [2] with underlying G-convex spaces and Theorem 3.3 of [23] for a FC-space setting are special cases of Theorem 4.1, since these spaces are also GFC-spaces. The proofs in [2, 23] are based on extensions of Ky Fan's matching theorem to these spaces. In our proof we apply Corollary 3.1, a consequence of our results in the Section 2 on continuous selections and in Section 3 on collectively fixed points. As far as we know this is the first time that selection theorems and collectively fixed point theorems are explicitly related to weakly KKM theorems.

(ii) If X and Y are topological spaces, A is a set, $T : X \to 2^Y$ is usc and $H : A \to 2^Y$ has closed values, then the set $\{x \in X : T(x) \cap H(a) \neq \emptyset\}$ is closed for each $a \in A$. The following example shows that the converse is not true.

Example 4.1 Let $X = [-2, 2]$, $Y = \mathbb{R}$, $A = \mathbb{N}$, $T : X \to 2^Y$ and $H : A \to 2^Y$ be defined by

$$T(x) = \begin{cases} [2,3] & \text{if } x = 1, \\ [0,1] & \text{otherwise,} \end{cases}$$

$$H(a) \equiv (0, 3).$$

Moreover, if for any $N = \{a_0, ..., a_n\} \in \langle A \rangle$, we define $\varphi_N : \Delta_n \to [0, 1]$ as the canonical projection on the first coordinate axis (among n + 1 axes), then $\{x \in X : T(x) \cap H(a) \neq \emptyset\}$ is closed for all $a \in A$. But $T(.)$ is not u.s.c and $H(.)$ is weakly T-KKM but not closed-valued. Consequently, applied to the special case where $A \equiv X$ and Y is a topological space, Theorem 4.1 improves Theorem 3.4 and 3.6 of [23].

Now we apply Theorem 4.1 to minimax inequalities of the Ky Fan type. Let X be a Hausdorff space, $(X, A, \{\varphi_N\})$ a GFC-space, Y a topological space, $T : X \to 2^Y$, $f : A \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 4.1 Let $\lambda \in \mathbb{R}$. f is called (λ, T, g)-GFC quasiconvex if for each $x \in X$ each $y \in T(x)$, each finite subset $N = \{a_0, ..., a_n\}$ of A and each $N_k = \{a_{i_0}, ..., a_{i_k}\} \subseteq N$ one has the implication

$$f(a_{i_j}, y) < \lambda \text{ for all } j = 0, ..., k \text{ implies } g(x', y) < \lambda \text{ for all } x' \in \varphi_N(\Delta_k).$$

For $\lambda \in \mathbb{R}$, define $\beta \in \mathbb{R}$ and $H_\lambda : A \to 2^Y$ by

$$\beta = \inf_{x \in X} \sup_{y \in T(x)} g(x, y),$$

$$H_\lambda(a) = \{y \in Y : f(a, y) \geq \lambda\}.$$

Lemma 4.1 *For $\lambda < \beta$, if f is (λ, T, g)-GFC quasiconvex then H_λ is a weakly T-KKM mapping.*

Proof Suppose to the contrary that there exist a finite subset $N = \{a_0, ..., a_n\}$ of A, $N_k = \{a_{i_0}, ..., a_{i_k}\} \subseteq N$ and $\bar{x} \in \varphi_N(\Delta_k)$ such that $T(\bar{x}) \cap H_\lambda(N_k) = \emptyset$. Then for all $y \in T(\bar{x})$ and $j = 0, ..., k$, $f(a_{i_j}, y) < \lambda$. By the (λ, T, g)-GFC quasiconvexity of f, one has $g(x', y) < \lambda$ for all $x' \in \varphi_N(\Delta_k)$. Then, $g(\bar{x}, y) < \lambda$ for all $y \in T(\bar{x})$. Hence $\sup_{y \in T(\bar{x})} g(\bar{x}, y) \leq \lambda$, contradicting the fact that $\lambda < \beta$. □

Theorem 4.2 *Assume that T and $f(a, \cdot)$ are u.s.c and f is (λ, T, g)-GFC quasiconvex for all $\lambda < \beta$ sufficiently close to β. Then the following statements hold*

(i) *if, in addition, X is compact then*

$\inf_{x \in X} \sup_{y \in T(x)} g(x, y) \leq \sup_{x \in X} \inf_{a \in A} \sup_{y \in T(x)} f(a, y);$

(ii) $\inf_{x \in X} \sup_{y \in T(x)} g(x, y) \leq \inf_{N \in \langle A \rangle} \sup_{x \in \varphi_N(\Delta_n)} \min_{a \in N} \sup_{y \in T(x)} f(a, y)$.

Proof Let $\lambda < \beta$ be arbitrary. By Lemma 4.1, H_λ is weakly T-KKM. Moreover, since $f(a, \cdot)$ is usc, H_λ has closed values. Therefore, the set $\{x \in X \mid T(x) \cap H_\lambda(a) \neq \emptyset\}$ is closed for all $a \in A$ (see Remark 8(ii)).

(i) According to the Theorem 4.1(i), we have an $\bar{x} \in X$ such that $T(\bar{x}) \cap H_\lambda(a) \neq \emptyset$ for each $a \in A$. Then

$$\lambda \leq \inf_{a \in A} \sup_{y \in T(\bar{x})} f(a, y)$$

and hence

$$\lambda \leq \sup_{x \in X} \inf_{a \in A} \sup_{y \in T(x)} f(a, y).$$

Since $\lambda < \beta$ is arbitrary, the statement is proved.

(ii) By Theorem 4.1(ii), for each $N \in \langle A \rangle$, there exists a point $\bar{x} \in \varphi_N(\Delta_n)$ such that $T(\bar{x}) \cap H_\lambda(a) \neq \emptyset$ for each $a \in N$. Consequently,

$$\lambda \leq \min_{a \in N} \sup_{y \in T(\bar{x})} f(a, y),$$

whence, for each $N \in \langle Y \rangle$,

$$\lambda \leq \sup_{x \in \varphi_N(\Delta_n)} \min_{a \in N} \sup_{y \in T(x)} f(a, y).$$

This implies that

$$\lambda \leq \inf_{N \in \langle A \rangle} \sup_{x \in \varphi_N(\Delta_n)} \min_{a \in N} \sup_{y \in T(x)} f(a, y).$$

Since $\lambda < \beta$ is arbitrary, we are done. □

References

1. Q. H. Ansari, J. C. Yao, A fixed point theorem and its application to a system of variational inequalities, *Bull. Austral Math. Soc.,* **59**(1999), 433-442.

2. M. Balaj, Weak G-KKM mappings, G-KKM property and minimax inequalities, *J. Math. Anal. Appl.,* **294**(2004), 237-245.

3. H. Ben-El-Mechaiekh, S. Chebbi, M. Flonzano, J. V. Llinares, Abstract convexity and fixed points, *J. Math. Anal. Appl.,* **222**(1998), 138-150.

4. F. E. Browder, The fixed point theory of multi-valued mappings in topological spaces, *Math. Ann.,*(1968), 283-301.

5. F. E. Browder, Coincidence theorems, minimax theorems and variational inequalities, *Contemporary Math.,* **26**(1984), 67-80.

6. X. P. Ding, J. Y. Park, Continuous selecion theorem, coinciddence theorem and generalized eqiulibrium in L-convex spaces, *Comput. Math. Appl.,* **44**(2002), 95-105.

7. X. P. Ding, J. Y. Park, Collectively fixed point theorem and abtract economy in G-convex spaces, *Numer. Funct. Anal. Optim.,* **23**(2002), 779-790.

8. X.P. Ding, Maximal element theorems in product FC-spaces and generalized games, *J. Math. Anal. Appl.,* **305**(2005), 29-42.

9. X. P. Ding, Continuous selection, Collectively fixed points and system of coincidence theorems in product topological spaces, *Acta Math. Sinica,* English series, **22**(2006), 1629-1638.

10. X. P. Ding, Collectively fixed points and a system of coincidence theorems in product FC-spaces and applications, *Nonlinear Anal.,* **66**(2007), 2604-2617.

11. N.X. Hai, P.Q. Khanh, N.H. Quan, Some existence theorems in nonlinear analysis for mappings on GFC-spaces and applications, *Nonlinear Anal.,* **71**(2019), 6170-6181.

12. C. D. Horvath, Some results on multi-valued mappings and inequalities without convexity, in: Nonlinear and Convex Analysis, B.L. Lin and S. Simons, Eds. (1987), 99-106.

13. C. D. Horvath, Extension and selection theorems in topological spaces with generalized convexity structure, *Ann. Fac. Sci. Toulouse,* **2**(1993), 353-369.

14. P. Q. Khanh, N. H. Quan, Intersection theorems, coincidence theorems and maximal-element theorems in GFC-spaces, *Optimization,* **59**(2010), 1-10.

15. P. Q. Khanh, N. H. Quan, General existence theorems, alternative theorems and applications to minimax problems, *Nonlinear Anal.,* **72**(2010), 2706-2715.

16. P. Q. Khanh, N. H. Quan, The solution existence of general inclusions using generalized KKM theorems with applications to minimax problems, J. Optimization Theory and Applications, to appear.

17. P. Q. Khanh, N. H. Quan, Generic stability and essential components of generalized KKM points and applications, submitted for publication.

18. P. Q. Khanh, N.H. Quan, J.-C. Yao, Generalized KKM type theorems in GFC-spaces and applications, *Nonlinear Anal.,* **71**(2009), 1227-1234.

19. K. Q. Lan, J. Webb, New fixed point theorems for a family of mappings and applications to problems on sets with convex sections, *Proc. Amer. Math. Soc.,* **126**(1998), 1127-1132.

20. E. Michael, Continuous selections I, *Ann. Math.,* **63**(1956), 361-382.

21. S. Park, Continuous selection theorems in generalized convex spaces, *Number Funct. Anal. Optim.,* **25**(1999), 567-583.

22. S. Park, H. Kim, Coincidence theorems for admissible multifunctions on generalized convex spaces, *J. Math. Anal. Appl.,* **197**(1996) 173-187.

23. G. S. Tang, Q. B. Zhang, C. Z. Cheng, W-G-F-KKM mapping, intersection theorems and minimax inequalities in FC-space, *J. Math. Anal. Appl.,* **334**(2007), 1481-1491.

24. E. Tarafdar, A fixed point theorem and equilibrium point of an abstract economy, *J. Math. Econom.,* **20**(1991), 211-218.

25. E. Tarafdar, Fixed point theorems in H-spaces and equilibrium points of abstract economies, *J. Austral. Math. Soc.,* series A **53**(1992), 252-260.

26. Z. T. Yu, L. J. Lin, Continuous selections and fixed point theorems, *Nonlinear Anal.,* **52**(2003), 445-455.